

Geometrical enhancement of the electric field: Application of fractional calculus in nanoplasmonics

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We developed an analytical approach, for a wave propagation in metal-dielectric nanostructures in the quasi-static limit. This consideration establishes a link between fractional geometry of the nanostructure and fractional integro-differentiation. The method is based on fractional calculus and permits to obtain analytical expressions for the electric field enhancement.

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I. INTRODUCTION

An interplay between the light and fractional nanostructures leads to a local giant enhancement of the electric field. This is one of the main effect of *nanoplasmonics* [1, 2] that concerns with the electromagnetic field on a nanoscale less then the wavelength inside a composite matter. There is a vast literature starting from seminal results on optics in fractal smoke [3] with further developing, where strong fluctuations of a local electromagnetic field are taken into account, see detailed discussion in [4] and recent reviews [1, 2, 5]. A suitable mechanism of these phenomena have been proposed in many studies [1, 5–7]), and it relates to the optical properties of metal nanoparticles, and, mainly, is due to their ability to produce giant and highly localized electromagnetic fields inside composite nanostructure. We are speaking about a *localized* surface plasmon resonance (SPR) (see, for example Ref. [1, 6]). Localized surface plasmons are charge density oscillations confined to conducting (metallic) nanostructures. A strong enhancement of the electromagnetic field is possible if its frequency is near the frequency of these oscillations. An analogous phenomenon, also known in optics, is a so-called enhanced Raman scattering. This remarkable phenomenon has been firstly observed more than thirty years ago, see *e.g.*, [5, 7] and properties of the Raman spectroscopy, has been used for a variety of applications. Theoretical explanation follow these experiments [7] and still attract much attention [1, 2, 6, 7].

In this paper we suggest an analytical consideration, based on fractional calculus, purposely to obtain an analytical expression for the electric field enhancement in nanoplasmonics. This approach establishes a relation between fractal geometry of the nanostructure and fractional integro-differentiation. It is based on averaging an extensive physical value expressed by means of a smooth function over a fractal set that leads to fractional integration [8]. We suggest a coarse graining procedure of the electric field in the Maxwell equation to treat a charge density term, which diverges everywhere in the case of fractional metal-dielectric composite media. This smoothing procedure makes possibility to obtain an equation for an electric field in a closed form. Our approach is based on a seminal result [9], where this scheme has

been suggested for a Cantor set. In its eventual form, it has been presented in Ref. [10] (in Ch. 5). The main idea of embedding or filtering matter inside a fractal is the construction of a convolution integral

$$M(x) = W(x) \star f(x) = \int_0^x W(x-y)f(y)dy, \quad (1)$$

where the function $W(x)$ obeys the following scaling relation $W(x) = \frac{1}{a}W(bx)$ with a solution $W(x) = t^\alpha A(x)$, where $\alpha = \frac{\ln a}{\ln b}$ and $A(x) = A(bx)$ is a log-periodic function with a period $\ln b$. When this scaling corresponds to a Cantor set with $a = 2$, and $A(x)$ is defined explicitly, one performs averaging over this period [10] and obtains the convolution integral in the form of a fractional integral

$$\langle M(x) \rangle = \frac{\mathcal{V}(\alpha)}{\Gamma(\alpha)} \int_0^x (x-x')^{\alpha-1} f(x') dx', \quad (2)$$

where $\mathcal{V}(\alpha) = \frac{2^{-1+\alpha/2}}{\ln 2}$ and $\Gamma(\alpha)$ is a gamma function. One should recognize that the coarse graining procedure due to the averaging over the period $\ln b$ is important for the application of fractional calculus, and “a bond between fractal geometry and fractional calculus” can be established on some coarse graining geometry [10]. This mathematical construction is relevant to study of electrostatics of real composite structures in the framework of coarse-graining Maxwell’s equation. The main objective of the preset research is an *analytical* derivation of the electric field in a composite dielectric, which is subject to an external high-frequency electric field at the conditions when the quasistatic limit is valid.

II. FRACTIONAL CALCULUS BRIEFLY

Extended reviews of fractional calculus can be found *e.g.*, in [11–13]. Fractional integration of the order of α is defined by the operator

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1} dy,$$

where $\alpha > 0$, $x > a$. Fractional derivation was developed as a generalization of integer order derivatives and is defined as the inverse operation to the fractional integral.

Therefore, the fractional derivative is defined as the inverse operator to ${}_a I_x^\alpha$, namely ${}_a D_x^\alpha f(x) = {}_a I_x^{-\alpha} f(x)$ and ${}_a I_x^\alpha = {}_a D_x^{-\alpha}$. Its explicit form is

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x f(y)(x-y)^{-1-\alpha} dy.$$

For arbitrary $\alpha > 0$ this integral diverges, and, as a result of this, a regularization procedure is introduced with two alternative definitions of ${}_a D_x^\alpha$. For an integer n defined as $n-1 < \alpha < n$, one obtains the Riemann-Liouville fractional derivative of the form

$${}^{RL}_a D_x^\alpha f(x) = \frac{d^n}{dx^n} {}_a I_x^{n-\alpha} f(x), \quad (3)$$

and fractional derivative in the Caputo form

$${}_a^C D_x^\alpha f(x) = {}_a I_x^{n-\alpha} \frac{d^n}{dx^n} f(x). \quad (4)$$

There is no constraint on the lower limit a . For example, when $a = 0$, one has ${}_0^{RL} D_x^\alpha x^\beta = \frac{x^{\beta-\alpha} \Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}$, $\alpha, \beta > 0$. This fractional derivation with the fixed low limit is also called the left fractional derivative. Another important property is $D^\alpha I^\beta = I^{\beta-\alpha}$, where other indexes are omitted for brevity's sake. A convolution rule for the Laplace transform for $0 < \alpha < 1$

$$\mathcal{L}[{}_x I^\alpha f(x)] = s^{-\alpha} \tilde{f}(s) \quad (5)$$

is commonly used in fractional calculus as well.

III. QUASI-STATIC ELECTROMAGNETIC FIELD

We consider a sample of a size R which consists of a fractal metal nanosystem embedded in a dielectric host medium. The system is subject to an external electromagnetic field $\mathbf{E}_0(t)$ at an optical frequency ω , and we assume that the fractal-host inhomogeneities are much smaller than the light wavelength $\lambda = 2\pi c/\omega$: $l_0 \ll \lambda$, where c is the light speed and l_0 is a minimal size of the self-similarity of the fractal volume. In this case a quasi-static approximation is valid [14]. Therefore, the nanosystem contains a metallic fractal with a fractal volume $V_D \sim R^D$ and the permittivity $\varepsilon_m(\omega)$, which depends on optical frequency ω , while the dielectric host has the volume V_h and the permittivity ε_d . Following Refs. [6, 7], the wave magnetic component is not considered, since the magnetic permeability $\mu = 1$ in both media. The electric component satisfies the Maxwell equation in the Fourier frequency domain

$$\nabla \cdot [\varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)] = 0. \quad (6)$$

The permittivity can be expressed by means of a characteristic function $\chi(\mathbf{r})$ [6–8] in the form $\varepsilon(\mathbf{r}, \omega) = \varepsilon_m(\omega)\chi(\mathbf{r}) + \varepsilon_d[1 - \chi(\mathbf{r})]$, where the characteristic function

inside the fractal is $\chi(\mathbf{r}) = 1$, $\mathbf{r}(x, y, z) \in V_D$, while inside the dielectric host it reads $\chi(\mathbf{r}) = 0$, $\mathbf{r}(x, y, z) \in V_h$. At the boundaries the electric field consists of the external field and an induced field due to polarization of the fractal nanostructure. Splitting the electric field into the two components $\mathbf{E}(\mathbf{r}) = \tilde{\mathbf{E}}(\mathbf{r}) + \mathbf{E}_1(\mathbf{r})$, we looking for the reaction of the nanosystem on the external field, where $\tilde{\mathbf{E}}(\mathbf{r})$ is the electric field induced by \mathbf{E}_0 in the homogeneous nanosystem, when $\chi(\mathbf{r}) = 0$ for $\forall \mathbf{r}$, or $\chi(\mathbf{r}) = 1$ for $\forall \mathbf{r}$ and $\nabla \tilde{\mathbf{E}}(\mathbf{r}) = 0$. Here $\mathbf{E}_1(\mathbf{r})$ results from inhomogeneity of the nanostructure. Thus Eq. (6) can be rewritten in the form

$$\begin{aligned} \chi(\mathbf{r}) \nabla \cdot \mathbf{E}_1(\mathbf{r}) &= q(\omega) \nabla \cdot \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) \cdot \nabla \chi(\mathbf{r}) \\ &= -\tilde{\mathbf{E}}(\mathbf{r}) \cdot \nabla \chi(\mathbf{r}), \end{aligned} \quad (7)$$

where $q(\omega) = \frac{\varepsilon_d}{\varepsilon_d - \varepsilon_m(\omega)}$ is a spectral parameter [15]. The boundary condition for the normal component of the dielectric shift is $\mathbf{D}_n(\mathbf{r} \in S) = \mathbf{E}_n(\mathbf{r} \in S)$, where S means the sample boundaries.

The main complication of treating Eq. (7) is the polarization charge density term $\sim \nabla \chi(\mathbf{r})$. Since the characteristic function $\chi(\mathbf{r})$ is discontinuous, $\nabla \chi(\mathbf{r})$ diverges everywhere. Therefore, the next step of our consideration is coarse-graining the electric field, which is averaging Maxwell's Eq. (7). This procedure relates to integration with the characteristic function $\int \chi(\mathbf{r}) \nabla \mathbf{E}_1(\mathbf{r}) dV$ and evaluation of the integral $\int \nabla \chi(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) dV$. To that end, let us consider a spherical volume of the radius r , such that $l_0 \ll r \ll \lambda$. At this scale the magnetic component is not important and cannot be considered. In the sequel we will work with dimensionless variable $r/l_0 \rightarrow r$. The electric field does not change at this scaling. A fractal mass inside the volume is $\mathcal{M}(r) \sim r^D$, where $0 < D < 3$. Therefore, an average density of the metallic phase is of the order of r^{D-3} . For the filtering mass inside the fractal, we consider convolution of Eq. (2). We also, reasonably, suppose that the random fractal composite is isotropic, *i.e.*, the (averaged) similarity exponents coincides along all directions. This yields that one takes into account only radius r and any changes in the inclination and azimuth angle directions can be neglected. Then, we have for the divergence

$$\nabla \cdot \mathbf{E}_1(\mathbf{r}) = \nabla_r E_{r,1}(r) \equiv \nabla_r E_1(r). \quad (8)$$

Now, filtering inside the fractal due to characteristic function depends only on the radius r , and this yields the integrations [22]

$$\begin{aligned} \frac{1}{4\pi} \int \chi(\mathbf{r}) \nabla_r E_1(r) dV &= \int_0^r \chi(r') [\nabla_r E_1(r')] r'^2 dr' \\ &= \int_0^r \sum_{r_j \in V_D} \delta(r' - r_j) G'(r') dr' \end{aligned} \quad (9)$$

Here we define $G(r) = r^2 E_1(r)$ and $G'(r) \equiv \frac{d}{dr} G$. One obtains the integration of the electric field with the fractal density $\sum_{r_j \in V_D} \delta(r' - r_j)$. The latter corresponds to the fractal volume

$$\mu(r) = r^D = \int_0^r \sum_{r_j \in V_D} \delta(r' - r_j) r'^2 dr'. \quad (10)$$

Following Ref. [17] (Theorem 3.1), we obtain

$$\begin{aligned} \int_0^r G'(r') d\mu(r') &\sim \frac{1}{\Gamma(D-2)} \int_0^r (r-r')^{D-3} G'(r') dr' \\ &\equiv {}_0I_r^{D-2} G'(r). \end{aligned} \quad (11)$$

Therefore, we consider the integration in Eq. (9) as the convolution integral with the averaged fractal density $(r-r')^{D-3}$.

Now we estimate the integral $\int_0^r E(r') \nabla_{r'} \chi(r') r'^2 dr'$. The fractal dust V_D at the N th step of the construction consists of balls B_N with the radius Δ_N . For example, $\Delta_N \sim l_0$. In the limiting case one obtains $V_D = \lim_{N \rightarrow \infty} \bigcup B_N$ [18]. The characteristic function for every ball is $\chi(\Delta_N) = \Theta(r-r_j) - \Theta(r-r_j-\Delta_N)$. Differentiation of the characteristic function on the intervals $[r_j, r_j+\Delta_N]$ yields $\nabla_r \chi(\Delta_N) = \delta(r-r_j) - \delta(r-r_j-\Delta_N)$. Therefore, for any interval Δ_N and at $r = r_j$, integration with the electric field yields

$${}_rI_{r+\Delta_N}^1 E(r) r^2 \nabla_r \chi(r) = E(r) r^2 - E(r+\Delta_N)(r+\Delta_N)^2.$$

This expression is not zero in the limit $\Delta_N \rightarrow 0$. Let $E(r_j)$ is the electric field outside the ball B_N and $E(r_j+\Delta_N)$ denotes the internal electric field. The relation between them, due to Eq. (8.2) in Ref. [14] for polarization of a dielectric ball, is

$$E(r_j+\Delta_N) = \frac{3\varepsilon_d}{\varepsilon_m(\omega) + 2\varepsilon_d} E(r_j).$$

Therefore, the shift for the electric field in the limit $\Delta_N \rightarrow 0$ is

$$E(r_j) - E(r_j+0) = E(r_j) \frac{\varepsilon_m(\omega) - \varepsilon_d}{\varepsilon_m(\omega) + 2\varepsilon_d}. \quad (12)$$

Finally, integration of the polarization charge term yields

$$\begin{aligned} \int_0^r E(r') \nabla_{r'} \chi(r') r'^2 dr' &\equiv {}_0I_r^1 E(r) r^2 \nabla_r \chi(r) \\ &= \frac{\varepsilon_m(\omega) - \varepsilon_d}{\varepsilon_m(\omega) + 2\varepsilon_d} \sum_{r_j \in V_D} \int_0^r E(r') r'^2 \delta(r' - r_j) dr'. \end{aligned} \quad (13)$$

Again, one obtains the integration of the electric field with the fractal density $\sum_{r_j \in V_D} \delta(r' - r_j)$. This corresponds to the fractal volume (10), and, hence, we consider the integration in Eq. (13) as the convolution integral of Eq. (11) with the averaged fractal density $(r-r')^{D-3}$. Eventually, we obtain for the polarization charge term in Eq. (7)

$$\begin{aligned} \int_0^r E(r') \nabla_{r'} \chi(r') r'^2 dr' &\sim \\ -\frac{p(\omega)}{\Gamma(D-2)} \int_0^r (r-r')^{D-3} E(r') r'^2 dr' & \\ \equiv {}_0I_r^{D-2} [G(r) + \tilde{E}_0 r^2]. \end{aligned} \quad (14)$$

Here $\tilde{E}_0 = \tilde{\mathbf{E}} \cdot \hat{\mathbf{r}}$ is a projection of the external electric field on the radial direction inside the chosen spherical volume of the radius r and

$$p(\omega) = \frac{\varepsilon_d - \varepsilon_m(\omega)}{\varepsilon_m(\omega) + 2\varepsilon_d}. \quad (15)$$

Taking all these arguments and results in Eqs. (11) and (14), one presents Eq. (7) in the coarse-graining form

$$\begin{aligned} {}_0I_r^{D-2} G'(r) - q(\omega) {}_0I_r^1 G'(r) &- p(\omega) {}_0I_r^{D-2} G(r) \\ &= \frac{2p(\omega) \tilde{E}_0}{\Gamma(D+1)} r^D. \end{aligned} \quad (16)$$

The Laplace transform can be applied to Eq. (16). Since $G(r=0) = G'(r=0) = 0$ (the electric field of fractal charge density diverges slowly than $\frac{1}{r^2}$ [8, 16]), one obtains for $\tilde{G}(s) = \hat{\mathcal{L}}[G(r)]$ due to Eq. (5):

$$\tilde{G}(s) = \frac{2p(\omega) \tilde{E}_0}{\varepsilon_d s^3} \cdot \frac{1}{s - q(\omega) s^{D-2} - p(\omega)}. \quad (17)$$

Note that the second term in Eq. (16) is $q(\omega) G(r)$.

Before arriving at the main result, let us consider the limiting cases. For $\varepsilon_m(\omega) = \varepsilon_d$ one obtains $q(\omega) = \infty$ and $p(\omega) = 0$. This yields $E_1(r) = 0$, and the solution for the electric field is $E(r) = \tilde{E}_0 \equiv \tilde{E}_r$. Another limiting case is $|\varepsilon_m(\omega)| \rightarrow \infty$. In this case $q(\omega) = 0$ and $p(\omega) = -1$, thus $E_1 \sim \tilde{E}_0$. Important result here is that permittivity of the mixture is approximately ε_d that corresponds to the well known result in Ref. [14] [see Eq. (9.7)].

IV. SURFACE PLASMON RESONANCE

The most interesting case is the SPR, when $\text{Re}[\varepsilon(\omega)] = -2\varepsilon_d$. This also known as the Fröhlich resonance [19, 20]. The permittivity of the fractal metallic nanostructure at the resonance condition is a complex value $\varepsilon_m(\omega) = \varepsilon_1 + i\varepsilon_2$, where $\varepsilon_2/\varepsilon_1 \ll 1$ and is described by classical Drude formula[23]. At a small detuning from the resonance, when $\text{Re}[\varepsilon(\omega)] = -2\varepsilon_d + \varepsilon_2$ that corresponds to the width of the resonance in the frequency domain, $p(\omega)$ reaches the maximal values, that yields

$$\begin{aligned} p(\omega) &= -1 + \frac{3\varepsilon_d}{2\varepsilon_2} (1-i) \approx \frac{3\varepsilon_d}{2\varepsilon_2} (1-i), \\ q(\omega) &= \frac{1}{3} - \frac{\varepsilon_2}{9\varepsilon_d} (1-i) \approx \frac{1}{3}. \end{aligned} \quad (18)$$

These expressions are inserted in Eq. (17). Before carrying out the inverse Laplace transform $\hat{\mathcal{L}}^{-1}[\tilde{G}(s)]$, it is reasonable to simplify the second denominator. We recast Eq. (17) in the form

$$\tilde{G}(s) = 2p(\omega) \tilde{E}_0 \sum_{k=0}^{\infty} \frac{[q(\omega) s^{D-2} + p(\omega)]^k}{s^{k+4}}. \quad (19)$$

We take into account that for the scale $r \gg 1$, the Laplace parameter is small $s \ll 1$, and the binomial becomes approximately a monomial. For $D < 2$, the first term can be left, while for $D \geq 2$ the second term $p(\omega)$ is dominant. To keep the first term for $D < 2$, we have $(1/s) \sim r \gg (3\varepsilon_d/2\varepsilon_2)^{\frac{1}{2-D}}$. But this condition violates

another restriction for $r \ll \lambda/l_0$, and $p(\omega)$ is the most important term in the binomial, and cannot be omitted. The Laplace inversion of Eq. (19) can be performed using an expression for the Mittag-Leffler function [21]

$$\mathcal{E}_{(\nu,\beta)}(zr^\nu) = \frac{r^{1-\beta}}{2\pi i} \int_{\mathcal{C}} \frac{s^{\nu-\beta} e^{sr}}{s^\nu - z} ds = \frac{r^{1-\beta}}{2\pi} \int_{\mathcal{C}} e^{sr} \sum_{k=0}^{\infty} \frac{z^k}{s^{\nu k + \beta}} ds = \sum_{k=0}^{\infty} \frac{[zr^\nu]^k}{\Gamma(\nu k + \beta)}, \quad (20)$$

where \mathcal{C} is a suitable contour of integration, starting and finishing at $-\infty$ and $\nu, \beta > 0$. Comparing Eqs. (19) and (20), $\beta = 4$ and $\nu = 1$, one obtains for the electric field

$$E_1(r) = 2p(\omega) \tilde{E}_0 r \mathcal{E}_{(1,4)}[p(\omega)r]. \quad (21)$$

Since the argument of the Mittag-Leffler function is large, its asymptotic behavior is [21]

$$E_1(r) \sim \frac{2\tilde{E}_0 r}{p^2(\omega)} e^{p(\omega)r} \propto \frac{E_0}{\varepsilon_d} \exp \left[r \frac{3\varepsilon_d}{2\varepsilon_2} (1-i) \right]. \quad (22)$$

Eventually, we arrived at the exponential (geometrical) enhancement and giant oscillations of the respond electric field due to the fractal geometry of the metal-dielectric composite. This result is valid for $r \geq 1$. At the same condition one can also choose the SPR at $\text{Re} \varepsilon_m(\omega) = -2\varepsilon_d - \eta\varepsilon_2$, where η is defined from the maximum enhancement of the electric field. This yields $q(\omega) \approx 1/3$ and $p(\omega) \approx \varepsilon_d/\varepsilon_2$. Thus the first term in the denominator in Eq. (17) can be neglected and the latter can be recast in the form

$$\tilde{G}(s) = -\frac{2p(\omega)\tilde{E}_0}{q(\omega)} \sum_{k=0}^{\infty} \frac{[-p(\omega)/q(\omega)]^k}{s^{(k+1)(D-2)+3}}. \quad (23)$$

This yields the following expression for the electric field

$$E_1(r) = \frac{2p(\omega)\tilde{E}_0 r^{D-2}}{q(\omega)} \mathcal{E}_{(D-2,3)}[-p(\omega)r^{D-2}/q(\omega)]. \quad (24)$$

If η is chosen such that $\arg[\frac{-p(\omega)}{q(\omega)}] < \frac{(D-2)\pi}{2}$, the asymptotic behavior of the Mittag-Leffler function is exponential [21] that leads to the giant geometrical enhancement $E_1(r) \sim \exp \left[\left(\frac{3\varepsilon_d}{\varepsilon_2} \right)^{\frac{1}{D-2}} r \right]$ that, obviously, takes place at the condition $D > 2$.

When the resonance is exact and the detuning from the resonance is zero, then $\text{Re}[p(\omega)] = -1$. The arguments of the Mittag-Leffler functions in both equations (21) and (24) do not correspond to the exponential asymptotic behavior. In this case the geometrical enhancement of the electric field is absent, and the linear (“classical”) enhancement [20] of the electric field takes place due to the imaginary part of the pre-factor: $\text{Im} p(\omega) \sim \frac{\varepsilon_d}{\varepsilon_2} \gg 1$.

V. DISCUSSION ON THE GEOMETRICAL ENHANCEMENT OF THE ELECTRIC FIELD

The obtained expressions in Eqs. (19) and (21) (together with developing the coarse grained Maxwell equation in the form of Eqs. (16) and (17)) contain the main physical result. As seen, the resulting (enhanced) electric field depends on the parameter $p(\omega)$, which is the pre-factor and the argument of the Mittag-Leffler function (when $r \sim 1$) in Eqs. (21) and (24), and defined in Eq. (15). This parameter is absorption efficiency in the electrostatic (quasi-static) approximation [20]. In our case, it describes polarization, and it is obtained in Eq. (13) under evaluation of fractal boundary conditions. One should recognize that the linear (classical [20]) enhancement of the electric field is always takes place, due to Eq. (13) that corresponds to the enhancement of the electric field by a small particle and that is reflected by the pre-factor in Eq. (19). For a fractal small composite of many particles the situation differs essentially. Here the geometrical enhancement of the electric field is due to the Mittag-Leffler function and it depends on the argument of the complex value of $p(\omega)$. The exponential (geometric) enhancement takes place when $|p(\omega)| \gg 1$ and $\text{Re} p(\omega) > 0$ as in Eq. (21), or $\text{Re}[-p(\omega)] > 0$ as in Eq. (24). These conditions are fulfilled for those frequencies ω that are in the vicinity of the SPR: $\text{Re} p(\omega) = -2\varepsilon_d + \Delta$, where $\Delta \sim \varepsilon_2 \ll \varepsilon_d$.

We have to admit that the obtained expressions in the exponential forms are the upper bound of the electric field enhancement. The enhanced E_1 does not exceed of the order of 10^8 (V/cm), otherwise the nonlinear effects become important that violates the linear quasi-static consideration. This restriction yields $r(\varepsilon_d/2\varepsilon_2) \leq 20$, which is a reasonable value for experimental realizations. Therefore, we have the light wavelength $\lambda \sim 10^{-4}$ cm, fractal inhomogeneity size $l_0 \sim (10^{-6} \div 10^{-5})$ cm, and $\varepsilon_d/\varepsilon_2 \sim \omega\tau \sim 5 \div 15$, where ω is the optical frequency, while τ is the relaxation time. The latter value determines l_0 , as well, which was introduced above as a minimum self-similarity size. Note that $\tau \leq \tau_s \equiv \frac{l_0}{v_F}$, where τ_s is the surface relaxation time, while v_F is the velocity on the Fermi surface (for free electrons). Therefore, from the condition $\omega\tau \gg 1$, one obtains $l_0 \gg \frac{v_F}{\omega} \sim 10^{-7}$ cm.

A. Geometrical enhancement out of the SPR

Equation (21) describes geometrical enhancement of the electric field out of the SPR, as well. For those frequencies ω that $p(\omega) > 1$, the argument of the Mittag-Leffler function is large, when $1 \ll r \ll \frac{\lambda}{l_0}$ that leads (roughly) to the exponential $E_1(r) \sim \tilde{E}_0 \exp[p(\omega)r]$. For example, when $\varepsilon_m(\omega) = -\varepsilon_d$, the small imaginary part of $\varepsilon_m(\omega)$ is not important, and the condition for the large asymptotics of the Mittag-Leffler function is fulfilled. One obtains $p(\omega) \approx 2$ and $E_1(r) \sim \tilde{E}_0 \exp[2r]$ that yields an enhancement of the order of 10^4 for $r > 5$.

VI. CONCLUSION

We developed an analytical approach, for description of the wave propagation in metal-dielectric nanostructures in the *quasi-static* limit. The method is based on fractional calculus and permits to obtain an analytical expressions for the electric field enhancement. This approach establishes a link between fractional geometry of the nanostructure and fractional integro-differentiation. An essential (geometrical) enhancement of the electric field is obtained for the surface plasmon resonance at a certain relation between permittivities of the host and fractal metallic nanostructure, when $\text{Re}\varepsilon_m(\omega) = -2\varepsilon_d$ with a suitable detuning.

Important part of the analysis is developing convolution integrals that makes it possible to treat the fractal structure. The initial Maxwell equation (6) is local, since $l_0/\lambda \ll 1$ and space heterogeneity is accounted locally by virtue of the characteristic function. An accurate treatment of the fractal boundaries and recasting the Maxwell equation in the form of the convolution integrals by accounting fractal properties of the composite, eventually, leads to the coarse graining equations, which already take into account the space heterogeneity and nonlocal nature of the electric field and polarized dipole charges inside the composite. Therefore, the heterogeneity, caused by the fractal geometry, is reflected by the convolution of the

averaged fractal density and the electric field, according Eqs.(11) and (14). It is necessary to admit that this transform from “local” quasi-electrostatics to the non-local, which takes into account space dispersion of permittivity $\varepsilon(r)$ is mathematically justified and rigorous enough [17]. The obtained convolution integrals are averaged values, since the fractal density r^{D-3} is the averaged characteristics of the fractal structure, and, eventually, it determines the space dispersion of the permittivity $\varepsilon(r)$ of the mixture.

Summarizing, we have to admit that observation of macroscopic Maxwell’s equations is related to averaging of microscopic equations [14]. This procedure for fractal composite media is not well defined so far, since, according fractal’s definition, averaging over any finite volume depends on the size of this volume itself [18]. The main idea to overcome this obstacle is to refuse the local properties of equations and obtain nonlocal coarse graining Maxwell’s equations, which are already averaged.

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 - [22] In Cartesian’s coordinates it corresponds to the fractional Riemann-Liouville integral with the elementary fractional volume [12, 16] $dV_D = \frac{|xyz|^{D/3-1}}{\Gamma^3(D/3)} dx dy dz$. In the spherical coordinates, which corresponds to the Reisz definition of the fractional integral, the elementary fractional volume is $dV_D = \frac{2^{3-D} \Gamma(3/2)}{\Gamma(D/2)} |\mathbf{r}|^{D-3} r^2 dr \sin \theta d\theta d\phi$.
 - [23] The dependence of the permittivity of the fractal metallic nanostructure on the frequency of the external electric field is described by Drude formula (see *e.g.* [14, 20]). For a spherical volume it reads $\varepsilon_m(\omega) = \varepsilon_0 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$, where ω_p is a so-called plasma frequency, ε_0 is a high-frequency lattice dielectric constant, while the attenuation coefficient γ is small in comparison with the resonant frequency. Therefore, $\varepsilon_m(\omega) = \varepsilon_1 + i\varepsilon_2$, where $\varepsilon_1 = \text{Re}[\varepsilon_m(\omega)] = \varepsilon_0 - \omega_p^2/\omega^2$ and $\varepsilon_2 = \text{Im}[\varepsilon_m(\omega)] = \gamma\omega_p^2/\omega^3$.